P: ISSN No. 0976-8602 RNI No. UPENG/2012/42622 VOL.-9, ISSUE-2, April 2020

E: ISSN No. 2349-9443 Asian Resonance

Established The Relationship Among The Lattice Structure

Paper Submission: 20 /04/2020, Date of Acceptance: 29/04/2020, Date of Publication: 30/04/2020

Abstract

Lattices are ubiquitous in mathematics. The beauty and simplicity of its abstruction and its ability to tie together seemingly unrelated pieces of Mathematics . I have made a graph that shows different types of lattices and their relationships. I think it may help the study of lattice theory .Lattices can be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent Lattice theory draws on both order theory and universal algebra semilattices include lattices which in turn include Heyting and Boolean algebras These "Lattice – structures admit all order theoretic as well as algebraic approach.

Keywords: Poset, Lattice, Hessa Diagram, Totally Ordered Set, Bounded Lattice, Distributive Lattice, Symmetric Lattice, Properties of Lattices.

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Introduction

Lattice theory, As an independent branch of mathematics has had a somewhat stormy existence during mid nineteenth century. It is studied in the mathematical disciplines of order theory and abstract algebra. It consists of a partially ordered set in which every two elements have a unique supremum (leastupper bound) denoted by $x \vee y$ and an unique infimum (greatest lower bound) denoted by $x \wedge y$. A lattice L is complete when each of its subsets X has a l.u.b and a g.l.b in L. Evidently the dual of any lattice is a lattice, and the dual of any complete lattice is a complete lattice, with meets and joins interchanged any finite lattice or lattice of finite length is complete.

The concept of a lattice (Dual gruppe) was first studied in depth by Dedekind. [(I), pp-113-14].

Aim of the Study

To establish the relations among the different types of Lattices.

Definition: Boolean Algebras

A Boolean lattice was defined as a complemented distributive lattice by definition, such a lattice must contains universal bound 0 and 1

$X \wedge X' = 0$, $X \vee X' = 1$

$(X')' = X$, $(X \wedge Y)' = (X' \vee Y')$ and $(X \vee Y)' = X' \wedge Y'$

i.e, In any Boolean lattice , each element x has one and only one complement x'.

Theorem: 1. If every element in a lattice L has a unique complement a' and if:

$(a \wedge b)' = a' \vee b'$

$(a \vee b)' = a' \wedge b'$ holds.

Then, L is a Boolean lattice.

Proof: By (commutativity)

$a \wedge a' = 0$ and $a \vee a' = I$

imply $a' \wedge a = 0$ and $a' \vee a = I$;

Hence $(a')^a = a$ for all $a \in L$ .... (1)

We next show that ,

$b \geq a$ implies $(b \wedge a') \vee a = b$

Indeed setting $C = b \wedge a'$ $(b \geq a)$

$c \wedge a \leq a' \wedge a = 0$ is immediate

more over ,

$(a \wedge b)' = a' \vee b'$, $(a \vee b)' = a' \wedge b'$

And $(a') = a$ for all $a \in L$

$0 = (b' \vee a) \wedge (a' \wedge b) = [(b' \vee a) \vee a]' = (c \vee a)'$

Substituting back , we get

$0 = (c \vee a)' \wedge b$ on the other hand ,

$c \vee a = (b \wedge a') \vee a \le b \vee a = b$,

So, $(c \vee a)' \vee b \ge b' \vee b = I$

Combining, $(c \vee a)'$ is the complement of b, and

So, $b = (b')' = [(c \vee a)']' = c \vee a = (b \wedge a) \vee a$.

Ortholattices

An Ortholattice is a Lattice with consider nondistributive analogs of

Of Boolean algebra in which universal bounds and a binary operation $\mathbf{a} \rightarrow \mathbf{a}^{\perp}$

Satisfying:

1. $a \wedge a^{\perp} = 0$, $a \vee a^{\perp} = I$ for all a
2. $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$, $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$
3. $(a^{\perp})^{\perp} = a$

Trivially, any ortholattice is complemented and a distributive ortholattice is a Boolean algebra the most familiar non – distributive ortholattice is that of sub-spaces of a finite – dimensional we write $a \subset b$ (in words a commutes with b)

Orthomodular Lattice

An ortholattice which satisfices either (hence both) of the equivalent conditions $x \in y$ implies $y \subset x$ (commutativity) is symmetric is called an orthomodular Lattice .

Subalgebras

Any lattice L is isomorphic with the lattice L' of all its principal ideals. If L is finite, then these are the " subalgebras " of A = [L, F] relative to the binary join operation $a \vee b$ and the unary projection operations $\Psi_c : a \rightarrow a \wedge c$ .Hence any finite (complete) Lattice of all subalgebras of a suitable algebra.

Complemented Lattice

Let *L* be a bounded lattice with greatest element 1 and least element 0. Two elements *x* and *y* of *L* are complements of each other if and only if:

$$x \vee y = 1 \text{ and } x \wedge y = 0$$

In this case, we write $x^{\perp} = y$ and equivalently, $y^{\perp} = x$. A bounded lattice for which every element has a complement is called a complemented lattice. The corresponding unary operation over *L*, called complementation, introduces an analogue of logical negation into lattice theory. The complement is not necessarily unique, nor does it have a special status among all possible unary operations over *L*. A complemented lattice that is also distributive is a Boolean algebra. For a Boolean algebra, the complement of *x* is probably unique.

Heyting Algebras

Heyting algebras are the example of distributive lattices having at least some members lacking complements. Every element *x* of a Heyting algebra has, on the other hand, a pseudo-complement, also denoted $\neg x$. The pseudo-complement is the greatest element *y* such that $x \wedge y = 0$. If the pseudo-complement of every element of a Heyting algebra is in fact a complement, then the Heyting algebra is in fact a Boolean algebra.

A lattice is called relatively complemented if all its closed intervals are complemented.

Residuated Lattice

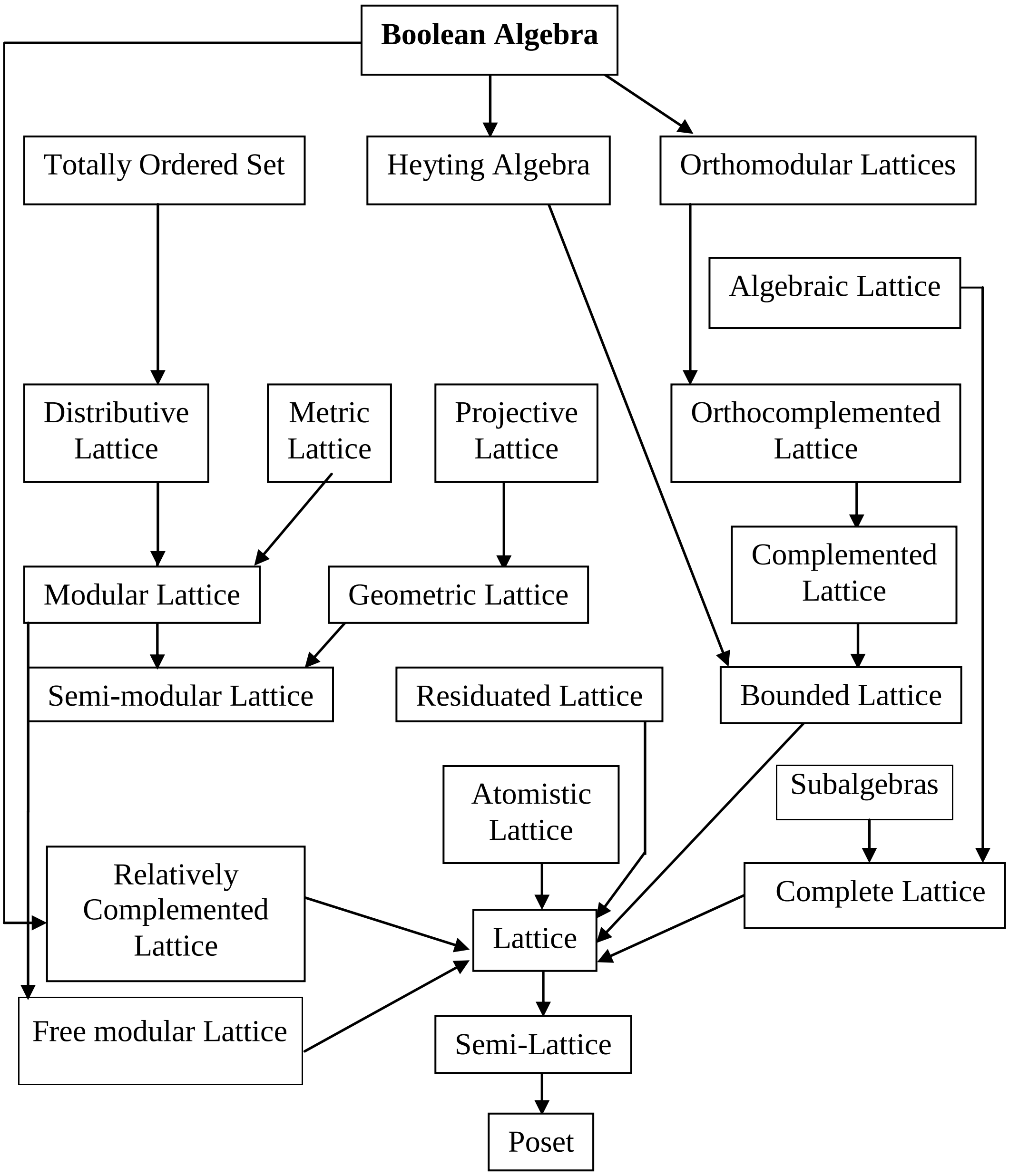
Let *L* be any po-groupoid. The right-residual $a : b$ of *a* by *b* is the largest *x* (if it exists) such that $bx \le a$; the left residual $a : b$ of *a* by *b* is the largest *y* such that $yb \le a$. A residuated lattice is an ℓ-groupoid *L* in which $a : b$ and $b : a$ exist for all $a, b \in L$.

Map of Lattices

Lattices can also be characterized as algebraic structures satisfying

Certain axiomatic identities that cause these are related to each other. The Hasse diagram below depicts the inclusion relationships among some important subclasses of lattice .This graphs shows different types of lattices and their common relationship I think it may help in studies of Lattices.

The concept of a lattices arises in order theory, a branch of Mathematics. The Hasse diagram below depicts the inclusion relationships among some important subclasses of lattices.



Proofs of the relationships in the map are inherited from the book: Introduction to lattice theory By Rutherford, Oliver and Boyd publication,

Conclusion

An important class of orthoposets wider than the Boolean algebras is the class of orthomodular lattices (Rf. [4]). It was proved in [1] that for every orthomodular lattice L there exists a compact Hausdorff closure (not necessarily topological) space L such that the orthomodular lattice of all closed subsets of L is isomorphic to L.

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